i.e.,

$$\int_{z_0}^{\infty}\frac{dz'\,Q_l(z')}{[K(z',z_1,z_2)]^{1/2}}-2\int_{z_a}^{1}\frac{dz'\,Q_l(z')}{[K(z',z_1,z_2)]^{1/2}}=[Q_l(z_1)]^2\cdots \quad (4.11)$$

for

$$z_1^2-1=\frac{4\mu^2}{s-4m^2}+\frac{4(M^2-m^2-\mu^2)^2}{(s-4m^2)(s-4m^2)}<0.$$

That this is so can be seen continuing in z_1 variable the identity (4.9). It should be emphasized that (4.11) is an identity derived by continuation in z_1; the validity of (4.11) does not depend on the assumption of unitarity.

V. CONCLUDING REMARKS

We have so far shown that certain amplitudes with complex singularities do not yield cuts as a direct consequence of the Regge continuation. The problem remains as to what mechanism can produce cuts if they are present. Amati *et al.*[2] have suggested that ordinary elastic unitarity in the crossed channel may yield cuts. But it is obviously insufficient to retain only elastic unitarity in calculating $\mathrm{Im}A(s,t)$. The optical theorem says $\mathrm{Im}A(s,t=0)\sim s\sigma^{\mathrm{total}}$, where σ^{total} is the total cross section, not just elastic cross section σ^{el}. Of course, $\sigma^{el}\sim 1/\ln s$, which simply expresses the shrinkage of the diffraction peak. It is not enough to calculate $\mathrm{Im}A(s,t)$ from elastic unitarity alone; one must include all the *inelastic* channels, which become available at large s. Thus,

$$\mathrm{Im}A(s,t)=\mathrm{Im}A^{(1)}(s,t)+\mathrm{Im}A^{(2)}(s,t),$$

where the first part comes from elastic, and the second from inelastic unitarity. It is perfectly consistent that

$$\mathrm{Im}A^{(1)}(s)\sim s^{\alpha(t)}/\ln s,$$

while $\mathrm{Im}A(s,t)\sim s^{\alpha(t)}$. The inelastic contribution just cancels the apparent cut coming from the elastic contribution. Thus, that argument is inconclusive.

It would be most satisfactory if we could say there are no cuts. However, the situation is inconclusive. If the cuts are present in some channel, they would be present in all channels with which this channel communicates if the sharing theorem continues to hold, and there is no reason to think that it would not. It may well happen that the strength of the branch cut might be greater for processes with prominent anomalous thresholds.

In concluding, we would like to emphasize that Regge amplitudes, even with complex singularities, are remarkably well behaved, and a calculation program based on them would thus probably circumvent to a considerable degree the complexities associated with complex singularities.

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A Mechanism for the Induction of Symmetries Among the Strong Interactions

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A model is constructed in which there are N equally massive vector mesons, which are self-consistent bound states of pairs of these same vector mesons. It is shown that N must be equal to the number of parameters of some compact, semisimple Lie group, and that the renormalized coupling constants must be proportional to the structure constants of the group.

THE strong interactions are known to exhibit isotopic spin symmetry, which is based on the group SU_2. There is also evidence that a further symmetry described by SU_3 exists, although this further symmetry is certainly much more approximate. In this paper, we raise the question of whether these symmetries might have a simple dynamical origin.

A phenomenological symmetry related to a Lie group is understood to mean two things. First, the mass spectrum of particles which have the same spin and parity should show a clustering which can be identified with the multiplet structure corresponding to representations of the group. Second, the S-matrix elements, and in particular, the renormalized coupling constants referring to different particles from the same multiplet should be related, approximately, through Clebsch-Gordan coefficients. It is clear that the origin of such a phenomenological symmetry could be established only through the development of a rather complete understanding of strong-interaction dynamics. Nevertheless, there is a possibility that the examination of simple models might help to clarify the problem. We are encouraged in this investigation by the observation that the ratios of the masses and coupling constants of particles which have the same spins and parities are somewhat easier to calculate than are other quantities.

We consider here a very simple model in which it is assumed that there are a number (N) of vector mesons which have the same mass. That is, it is assumed that the mass spectrum does exhibit a clustering effect. Then, with a number of additional assumptions that will be made clear, it will be shown that a Lie group can be associated with these particles. The essential feature of the model is that only these N vector mesons are introduced. They are all supposed to arise as self-consistent bound states of pairs of vector mesons, and the binding force is mediated by the exchange of single vector mesons. In this paper we shall assume that the N states are exactly degenerate.

We assume the spatial invariances usually associated with strong interactions, as well as invariance under charge conjugation. We may then represent the particles by real vector fields A_a, $a=1, \cdots, N$. The three-meson vertex has the form

$$g(F_{abc}\mathcal{O}+F_{abc}'\mathcal{O}')A_aA_bA_c\,, \tag{1}$$

where $\mathcal{O}$ and $\mathcal{O}'$ are certain functions of the momenta. Symmetry with respect to interchange of all three particles leads to the result that there are only two independent terms, and that F_{abc} and F_{abc}' are imaginary and antisymmetric, one in two, the other in all indices. We shall investigate only the case that $F_{abc}'\equiv F_{abc}$.

Two mesons can interact by exchanging another between them. The Born-approximation scattering amplitude is proportional to

$$V_{ab,cd}=F_{adr}F_{bcr}-F_{acr}F_{bdr} \tag{2}$$

in states which are antisymmetric in the spatial variables. To the extent that the sign of the force can be deduced from the "Coulombic" part of the interaction, it will be attractive between particles which have opposite charges, that is, when the expectation of $V_{ab,cd}$ is positive. In fact, the attraction is very singular at short distances, so we must cut it off at an effective mass value denoted by Λ.

The problem of actually solving the partial-wave dispersion relations following, for example, the method of Zachariasen and Zemach,[1] would be extremely complicated. Fortunately, we do not need to solve them; we only need to show that solutions having the desired properties exist. For this, it is sufficient to observe that if we have an attractive force, we can always adjust Λ so that a bound state of any desired mass—in particular, the vector-meson mass, which we take to be unity—could be obtained. The coupling constants for the bound state to the two vector mesons are obtained from the residue of the bound-state pole. We do not examine the different helicity amplitudes of the bound-state pole, because these are sensitive to the way the cutoff is introduced.

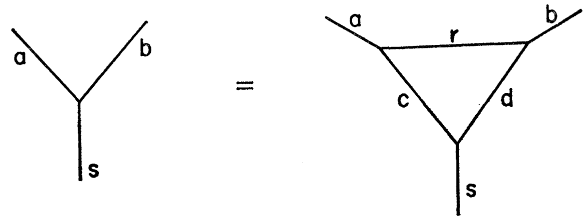

FIG. 1. A graphical representation of Eq. (3). Another interpretation of Eq. (3) is that the renormalized coupling constants are considered to be generated by the simplest irreducible vertex part, with the "bare-coupling constants" set equal to zero.

Since all the particles which are being bound together, and also all of the exchanged particles, have the same mass, it is clear that we can obtain N degenerate bound states only if V has N degenerate eigenvalues. Moreover, the F_{abs} must themselves be the eigenvectors;

$$\lambda F_{abs}=V_{ab,cd}F_{cds}\,, \tag{3}$$

where $\lambda>0$. Equation (3) can be pictured as in Figure 1. In addition, since the same residue will be obtained in each of these dynamically equivalent states, we can impose the normalization condition

$$F_{abr}F_{bas}=\delta_{rs}\,. \tag{4}$$

In terms of elementary wave-mechanical ideas, this can be interpreted as saying that in each vector-meson state, the total probability of all virtual pairs is unity. The actual value of the residue must be g^2, which leads to a second relation between g^2 and Λ.

There are $\frac{1}{2}N(N-3)$ antisymmetric eigenvectors $\psi_{ab}{}^i$ of $V_{ab,cd}$ which are orthogonal to F_{abs}. We denote their eigenvalues by λ_i and the degeneracy by d_i. There are, therefore, $\frac{1}{2}N(N-3)$ vector states in which the dynamics is identical to that in the N self-consistent states, except that the effective strength of the force is multiplied by a factor λ_i/λ. It seems plausible that, for given values of g^2 and Λ, the mass of a bound state would depend monotonically upon the eigenvalue of V. We must require that no other vector particles which have a lower mass (or even a slightly larger mass) than the N we started with should arise from the potential, because otherwise our model would not really be self-consistent; we would have to start over again and include the extra particles from the beginning. We, therefore, require that

$$\lambda>\lambda_i\,. \tag{5}$$

We shall see that this inequality is crucial to the determination of the F's.

It is clear that strong interactions in states with other spins and parities might also arise, and if so, these other states ought to be incorporated into the model.

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[1] F. Zachariasen and C. Zemach, Phys. Rev. **128**, 849 (1962).

We ignore such states here, with the understanding that the generality of our conclusions depends on the validity of the assumption that the influence of other states upon the vector particles can be adequately represented by the single adjustable parameter Λ.

We have now defined our model, and proceed to the determination of F_{abc} which satisfy the conditions (3), (4), and (5). We first observe that

$$N\lambda = F_{bas}V_{ab,cd}F_{cds} = 2F_{bas}F_{adr}F_{bcr}F_{cds}. \tag{6}$$

Now, we have from the definition of the trace that

$$\mathrm{Tr}\ V^2 = N\lambda^2 + \textstyle\sum_i d_i\lambda_i^2, \tag{7}$$

but in our case, the explicit form (2) gives, with the help of (4) and (6),

$$\mathrm{Tr}\ V^2 = 2N - N\lambda. \tag{8}$$

Therefore, it follows that

$$\lambda(\lambda+1) = 2 - N^{-1}\textstyle\sum_i d_i\lambda_i^2, \tag{9}$$

so that

$$\lambda \leq 1. \tag{10}$$

The equality can hold only if all λ_i vanish.

Under the orthogonal transformations $A_a' = A_bO_{ab}$ the equations (3) and (4) are covariant. For infinitesimal transformations $O_{ab} = \delta_{ab} + i\epsilon^\alpha G_{ab}^\alpha$, the F_{abc} transform according to

$$\begin{aligned} F_{abc}' &= F_{abc} + i\epsilon^\alpha f_{abc}^\alpha, \\ f_{abc}^\alpha &= F_{xbc}G_{xa}^\alpha + F_{axc}G_{xb}^\alpha + F_{abx}G_{xc}^\alpha. \end{aligned} \tag{11}$$

We shall now make a further assumption, that the interactions satisfy a nontrivial additive conservation law. For instance, we might assume that some of the particles have electric charges, and that charge is conserved at each vertex. This requirement gives additional information about the F_{abc}. It is not clear whether this additional information is necessary, but it greatly simplifies our analysis. If we assume such an additive conservation law, we can make gauge transformations of the first kind which leave the F_{abc} invariant. In the representation in which real A_a are used, these gauge transformations are just a special case of the orthogonal transformations introduced above. If we assume the existence of r independent additive conservation laws, (for example, charge and hypercharge) the F_{abc} will be invariant under an r parameter Abelian subgroup of O_N.

We denote by G_{ab}^A a generator of the Abelian gauge transformations. Then we have

$$F_{xbc}G_{xa}^A + F_{axc}G_{xb}^A + F_{abx}G_{xc}^A = 0. \tag{12}$$

If we multiply (12) by F_{bcd}, and use the fact that $G_{ab}^A = -G_{ba}^A$, we obtain

$$G_{cd}^A = V_{cd,ab}G_{ab}^A. \tag{13}$$

The generators G_{ab}^A are eigenvectors of V with a unit eigenvalue.

There are now two cases to be considered. If $F_{bac}G_{ab}^A = 0$, we have found another eigenvalue: $\lambda_A = 1$. Then, we see from (9) that $\lambda < 1$, so that $\lambda < \lambda_A$, and the condition (5) is violated. If, on the other hand, $F_{bac}G_{ab}^A \neq 0$ for some c, we have $\lambda = 1$, so that all λ_i vanish. The completeness of the eigenvectors of V then allows us to write

$$V_{ab,cd} = F_{abr}F_{dcr}, \tag{14}$$

which can be rewritten as

$$F_{abr}F_{cdr} + F_{bcr}F_{adr} + F_{car}F_{bdr} = 0. \tag{15}$$

Equations (15) and (4) are the necessary and sufficient conditions for the F_{abc} to be (apart from a factor i) the structure constants of a compact, semisimple Lie group. It is evident that the rank of the group must be at least as great as r. If we assume that the N particles cannot be divided into subsets which interact only among themselves, in some representation, the group must be simple.

It may be of interest to observe that examples of F_{abc} which satisfy (3) and (4) but violate (5) are provided by the Clebsch-Gordan coefficients of many group representations. For example, for SU_2, any representation corresponding to an odd-integral value of the isospin T can be used. In particular, one calculates for $T=3$, $\lambda=-1$; for $T=5$, $\lambda=+0.37$. For all T, in accordance with (13), in the state $T'=1$ we have $\lambda_1=1$.

Our conclusion is that a set of N degenerate vector mesons, which interact in accordance with the model set forth, can always be associated with a Lie group. Moreover, the association is necessarily with a particular representation of the group, the adjoint representation. Note that this association with the adjoint representation is exactly the same as that implied by a Yang-Mills type of theory, in which invariance under a simple group of transformations is assumed at the very beginning.[2,3]

Our model also leads to certain negative conclusions. We observe that λ, and hence, the cutoff Λ, is independent of the group. This suggests that it might be hard to use an extension of the self-consistent model, in which an attempt was made to calculate an *a priori* value of Λ from dynamical arguments, to eliminate certain groups from consideration. One must keep in mind the possibility that, in addition to certain additive quantum numbers which might be supposed to be exactly conserved, there might be other additive quantum numbers which arose out of the strong interactions themselves. However, it would certainly be very hard to determine the number of such quantum numbers from the self-consistency requirement.

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[2] C. N. Yang and R. Mills, Phys. Rev. **96**, 191 (1954).
[3] S. L. Glashow and M. Gell-Mann, Ann. Phys. (N.Y.) **15**, 437 (1961).